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The in!plane loading of rigid disc inclusion embedded in a crack

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Abstract

The paper examines the in!plane loading of a disc shaped rigid disc inclusion which is embedded in bonded contact with the plane surfaces of a penny!shaped crack[The mixed boundary value problem governing the elastostatic problem is reduced to the solution of a system of coupled integral equations\ which are solved numerically to determine results of engineering interest[These results include the in!plane sti}ness of the disc inclusion and the crack opening mode stress intensity factor at the boundary of the penny!shaped crack[

0[Introduction

The disc inclusion problem in the classical theory of elasticity is a particular simpli_cation of the general category of three!dimensional inhomogeneities[When the physical con_guration of the inhomogeneity allows its modelling as a disc inclusion\ the analysis of the inclusion problem can be considerably simpli_ed[Attention can be focused on the analysis of a variety of inclusion problems which are essentially mixed boundary value problems related to an elastic halfspace region[Elastostatic problems associated with disc inclusions have been successfully applied to examine a variety of problems of interest to the mechanics of multiphase composite materials and geomechanics[The investigations by Collins "0851#\ Keer "0854# and Kassir and Sih "0857# are the pioneering works in this area[Since these original developments\ the theory of a disc inclusion has been applied to a variety of situations involving anchor!type objects used in geomechanical applications[These studies have taken into consideration non!classical e}ects such as material anisotropy\ in~uence of bi!material regions\ ~exibility of the inclusion\ delaminations and cracking both within and exterior to the inclusion region and the interaction between the inclusion and externally placed loads[Accounts of these developments are given by Mura "0870\ 0877#\ and in the recent articles by Selvadurai et al["0889#\ and Selvadurai "0883a\ b#[

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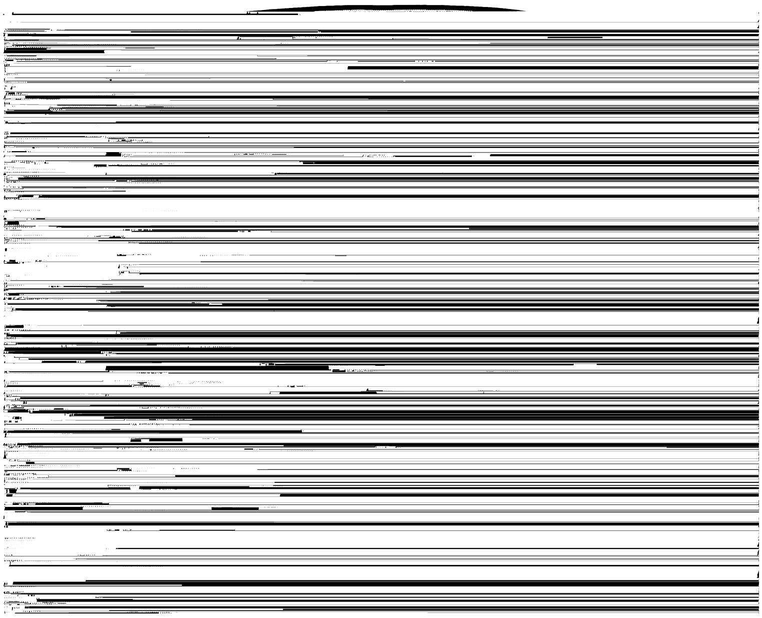


Fig[0[In!plane translation of a rigid disc inclusion embedded in a penny!shaped crack[

In this paper we examine the problem related to a disc!shaped rigid circular inclusion which is embedded at the centre of a penny!shaped crack[The problem may be visualized either as a situation where fracturing has extended beyond the boundary of a rigid disc inclusion or where an inclusion region is created by the injection of a cementitious material into a geological medium by hydraulic facturing "Fig[0#[The disc inclusion embedded in a crack is\ therefore\ an approximate analogue of the anchor region[In general the rigid anchor region can be subjected to various modes of deformation[The axial loading of a rigid disc anchor embedded in complete bonded contact with the faces of the penny!shaped crack was examined by Selvadurai "0878#[This result was extended by Selvadurai "0883b# to examine the case when the axial loading in the presence of delamination at one face of the inclusion[

In this study we extend the work to include the in!plane loading of the rigid circular disc inclusion for the particular case when the inclusion is in bonded contact with the faces of the penny!shaped crack[The in!plane loading of the inclusion is more consistent with situations where the anchorage is formed at orientations normal to a direction of minimum principal stress exerted\ for example\ by self weight stresses[Also\ the problem examined considers the case where the rigid disc anchor or inclusion is located\ in bonded contact\ at the centre of the penny!shaped crack[Such positioning is expected to produce an anchorage of highest compliance\ which is important to the assessment of the elastostatic e . ciency of the anchorage[The mixed boundary value problem resulting from the anchor "inclusion# crack interaction problem is reduced to the solution of a set of coupled integral equations which are solved in a numerical fashion[Numerical results are presented for the in!plane sti}ness of the inclusion and for the crack opening mode stress intensity factor at the boundary of the penny!shaped crack[

1[Governing equations

The associated asymmetric elastostatic boundary value problem can be formulated by employing the stress function techniques developed by Muki "0859#[The stress functions are governed by the di}erential equations

$$9^1 9^1 F"r\backslash u\backslash z\# \quad 9 \qquad "0a\#$$

$$9^1 C"r\backslash u\backslash z\# \quad 9 \qquad "0b\#$$

where

$$9^1 \quad \frac{1^1}{1r^1} \quad \frac{0}{r}\frac{1}{1r} \quad \frac{0}{r^1}\frac{1^1}{1u^1} \quad \frac{1^1}{1z^1} \qquad "1\#$$

is Laplace|s operator referred to the cylindrical polar coordinate system[The displacement and stress components in the elastic medium can be expressed in terms of the functions F"r\u\z# and C"r\u\z#[Considering a Hankel transform development of the governing equations we can show that the relevant solutions applicable to the region 9 z take the forms "3#

$$F"r\backslash u\backslash z\# \quad \cos u \int_9 \; j \; A"j\# \quad zB"j\# \; e^{\; jz} J_0"jr\# \, dj \qquad "2\#$$

$$C"r\backslash u\backslash z\# \quad \sin u \int_9 \; jC"j\# \; e^{\; jz} J_0 \; jr\# \, dj \qquad "3\#$$

where A"j#\ B"j# and C"jj

9

$$s_{uz}"r\backslash u\backslash z\# \quad \frac{\sin u}{r} \$ g_9 \quad j"j^1A"j\# \quad B"j\#"j^1z \quad 1nj\#\# e^{\quad jz} J_0"jr\# \, dj$$

$$g_9 \quad j^1C"j\#"jrJ_9"jr\# \quad J_0"jr\#\# e^{\quad jz} dj\% \qquad "8\#$$

$$s_{rz}"r\backslash u\backslash z\# \quad \frac{\cos u}{r} \$ g_9 \quad j" \quad j^1A"j\# \quad "1nj \quad j^1z\#B"j\#\#"jrJ_9"jr\# \quad J_0"jr\#\# e^{\quad jz} dj$$

$$g_9 \quad j^1C"j\#J_0"jr\# e^{\quad jz} dj\% \qquad "09\#$$

where G is the linear elastic shear modulus and n

$$\int_9 N"j\#J_0"jr\#\,dj \quad 9\wedge \quad 9 \quad r \quad a \qquad "08\#$$

$$\int_9 N"j\#J_0"jr\#\,dj \quad 9\wedge \quad b \quad r \qquad "19\#$$

$$\int_9 jL"j\#J_9"jr\#\,dj \quad 9\wedge \quad a \quad r \qquad "10\#$$

$$\int_9 jM"j\#J_1"jr\#\,dj \quad 9\wedge \quad a \quad r \qquad "11\#$$

$$\int_9 j\left\{L"j\# \quad M"j\# \quad \frac{1N"j\#}{"0 \quad 1n\#}\right\} J_0"jr\# \quad 9\wedge \quad a \quad r \quad b \qquad "12\#$$

where the functions L"j#\ M"j# and N"j# are related to the functions A"j#\ B"j# and C"j# according to

$$1j^2"0 \quad n\#A"j\# \quad 1nN"j\# \quad "0 \quad 1n\#"L"j\# \quad M"j\#\# \qquad "13\#$$

$$3j^1"0 \quad n\#B"j\# \quad 1N"j\# \quad L"j\# \quad M"j\# \qquad "14\#$$

$$1j^1C"j\# \quad L"j\# \quad M"j\# \qquad "15\#$$

Considering the integral equations "06# "12# we introduce the following representations

$$L"j\# \quad \int_9^a 8_0"t\#\cos"jt\#\,dt \quad \frac{8_0"a\#\sin"ja\#}{j} \quad \frac{0}{j}\int_9^a 8?"t\#\sin"jt\#\,dt \qquad "16\#$$

$$M"j\# \quad \int_9^a t8_1"t\#J_1"jt\#\,dt \qquad "17\#$$

with 8_0"9# 9 and the prime denotes the derivative of the function with respect to t[Substituting "16# and "17# into "10# and "11# we _nd that both equations are identically satis_ed[Substituting "16# into "06# we obtain

$$\int_9^r \frac{8_0"t\#\,dt}{"r^1 \quad t^1\#^{0:1}} \quad \frac{05Gd"0 \quad n\#}{"6 \quad 7n\#} \int_9 \left\{\frac{M"j\#}{"6 \quad 7n\#} \quad \frac{1"0 \quad 1n\#N"j\#}{"6 \quad 7n\#}\right\} J_9"jr\#\,dj\wedge \quad 9 \quad r \quad a \qquad "18\#$$

which is an integral equation of the Abel type\ the solution of which is given by

$$8_0"t\# \quad \frac{21Gd"0 \quad n\#}{"6 \quad 7n\#p} \quad \frac{1}{p"6 \quad 7n\#}\int_9 \quad M"j\# \quad 1"0 \quad 1n\#N"j\# \;\cos"jt\#\,dj\wedge \quad 9 \quad t \quad a \qquad "29\#$$

The eqn "12# can be written in the form

g

$$\int_9^a t\,8_1\text{"t\#}K\text{"r\textbackslash t\#}\,dt \quad \int_9 \frac{0}{\text{"6}\quad 7n\text{\#}}\text{"L"j\#}\quad 1\text{"0}\quad 1n\text{\#N"j\#\#}J_1\text{"jr\#}\,dj\,\wedge\quad 9\quad r\quad a \qquad \text{"37\#}$$

where

$$K\text{"r\textbackslash t\#}\quad \frac{1}{p\text{"rt\#}^1}\int_9^{\min\text{"r\textbackslash t\#}} \frac{s^3\,ds}{\text{"t}^1\quad s^1\text{\#"r}^1\quad s^1\text{\#\#}^{0:1}} \qquad \text{"38\#}$$

Also\ introducing the substitution

$$P\text{"s\#}\quad \int_s^a \frac{8_1\text{"t\#}\,dt}{t\text{"t}^1\quad s^1\text{\#}^{0:1}}\,\wedge\quad 9\quad s\quad a \qquad \text{"49\#}$$

and using the representations for L"j# and N"j# it can be shown that

$$P\text{"s\#}\quad \frac{p}{1s\text{"6}\quad 7n\text{\#}}\left\{\frac{8_0\text{"s\#}}{s}\quad \frac{0}{s^1}\int_9^s 8_0\text{"t\#}\,dt\right\}\quad \frac{1\text{"0}\quad 1n\text{\#}}{ps^2\text{"6}\quad 7n\text{\#}}\int_9^a \frac{F_0\text{"u\#}}{u}\log_e\left|\frac{u\quad s}{}\right.$$

integral equations is such that results of practical interest can be obtained only upon numerical solution of these equations[

3[Load displacement behaviour of the disc inclusion

The shear stress distribution at the disc inclusion elastic medium interfaces can be used to evaluate the in!plane load displacement relationship for the rigid disc inclusion[The shear traction on the plane z 9 in the x!direction is given by

$$T_x \quad "s_{rz} \cos u \quad s_{uz} \sin u\# \qquad \frac{0}{1}\int_9 j L"j\#J_9"j r\# \, dj \wedge \quad 9 \quad r \quad a \qquad "45\#$$

The resultant force T required to induce the in!plane displacement d is given by

$$T \qquad 1p \int_9^a r \, dr \int_9 j L"j\#J_9"j r\# \, dj \qquad "46\#$$

Since

$$\int_9 j"L\#"j\#J_9"j r\# \, dj \qquad \frac{8_0"a\#}{z\sqrt{a^1 \quad r^1}} \quad \int_r^a \frac{8_0^?"t\# \, dt}{z\sqrt{t^1 \quad r^1}} \qquad "47\#$$

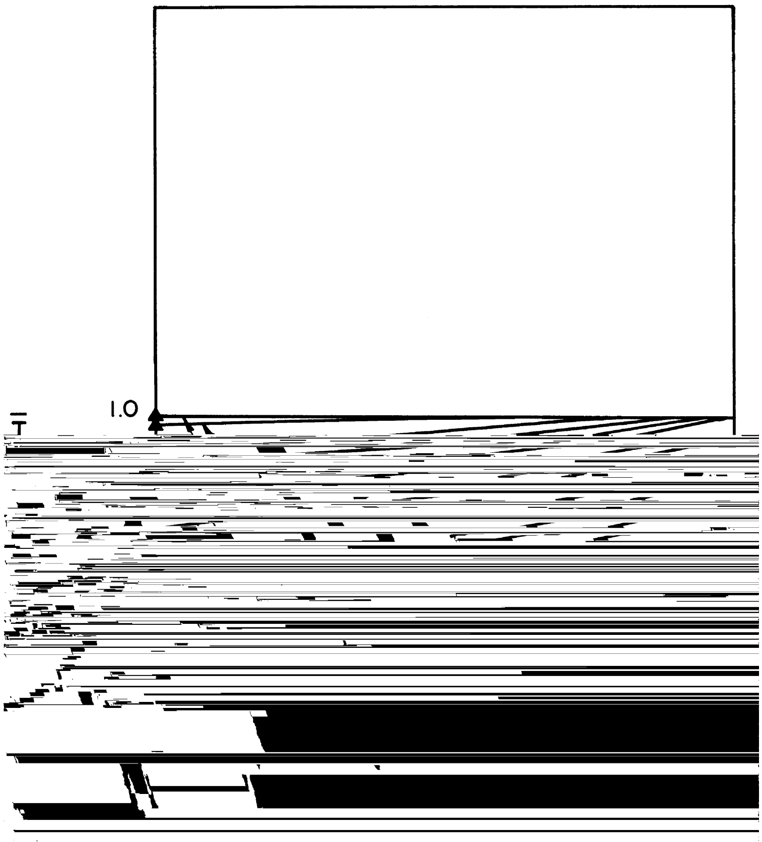
where the prime indicates the derivative with respect to t\ "46# can be reduced to the result

$$T \qquad 1p \int_9^a 8_0"t\# \, dt \qquad "48\#$$

4[Stress intensity factor at the crack tip

Since the state of deformation induced in the elastic medium as a result of the displacement of the inclusion is symmetric about the plane z 9\ the mode II and mode III stress intensity factors where the prime denotes the derivative with respect to the appropriate argument[The stress

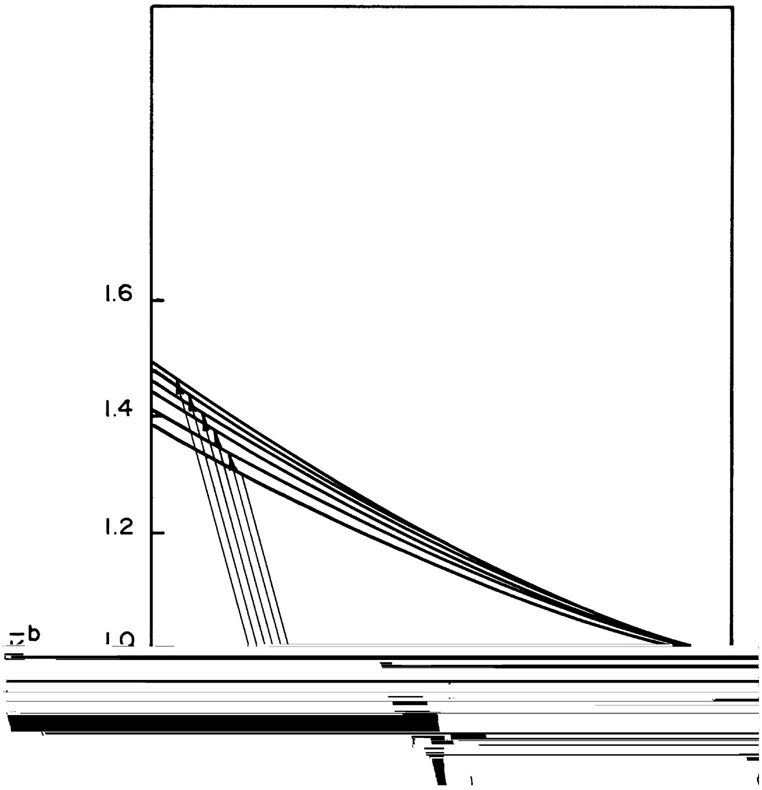
$$\frac{T}{53"0 \quad n\#GDa:"6 \quad 7n\#} \qquad \frac{0}{N_0} {}_j \, s^{1N_0}_{N_0} \; {}_0 \, X_j \qquad "57\#$$



Fig[1[The in~uence of the inclusion!crack aspect ration "a:b# on the in!plane sti}ness of the disc inclusion

$T T"6 7n# / 53GDa"0 n#%[

mixed boundary value problem referred to a halfspace region[It is shown that the mixed boundary value problem can be reduced to a system of coupled integral equations which can be solved by using a quadrature scheme\ to develop results of engineering interest[In the case when the inclusion is bonded to the surfaces of the crack\ the stress singularity at the boundary of the inclusion will exhibit an oscillatory form of a stress singularity[Studies conducted previously in connection with



Fig[2[The in~uence of the inclusion!crack aspect ration "a:b# on the mode I stress intensity factor at the crack tip r b

$$\$ K_I^b \quad \frac{K_I^b "6 \quad 7n\# p^1 b^{2:1}}{05GDa"0 \quad 1n\# \cos u} \%[$$

the axial loading of an inclusion embedded in a crack have shown that such local e}ects have very little in~uence on the overall responses such as the load displacement behaviour of the inclusion[The stress singularity at the crack tip is regular and the nonzero axial stress can be used to compute the mode I stress intensity factor[Furthermore\ the numerical results indicate that the in!plane sti}ness of the inclusion is not signi_cantly in~uenced by the extent of cracking in the plane of the inclusion[For all practical purposes\ the elastic solution can be conveniently computed by making use of the exact analytical result for the in!plane translation of a rigid punch which is embedded in bonded contact between two halfspace regions[

References