JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 10, 2008

ON A WINKLER LIGAMENT CONTACT BETWEEN A RIGID DISC AND AN ELASTIC HALFSPACE

A. P. S. SELVADURAI, T. SCARPAS AND N. KRINGOS

This paper presents a variational solution to the problem of the contact between an isotropic elastic halfspace and a rigid circular indentor, where the contact is achieved through a set of ligaments modeled by a continuously distributed layer of Winkler elements. The problem is of interest to the modeling of the ligament-type contact mechanics between a rigid cylinder and a substrate. The limiting solution for Boussinesq indentation is modi ed to take into consideration small but nite in uences of the elastic stiffness of the ligaments forming the interface layer.

1. Introduction

The mechanics of contact between a component and a substrate is of interest to many areas of ma
engineering and materials science. The classical de nition of adhesive contact between two mater
regions assumes the complete compatibility of displacements between the two regions. Other forms
nonclassical contacts include interacting surfaces that exhibit limited adhesion, frictional constraints
slip. The developments, both fundamental and applied, in this area are too numerous to cite indivi
ually. We mention [Duvaut and Lions 1976; Selvadurai 1979; 2003; 2007; Gladwell 1980; Haslinger and Janovský 1983; Johnson 1985; Ciardeveloping a plausible model that can determine the onset of

Plueddemann 1974; Anderson et al. 1977; de Lollis 1985; Pizzi and Mittal 1994; Mittal 1995]. Furthermore, depending on the nature of the interacting regions, the contact between the bodies in adhesive contact can in fact be induc
at discrete regions at the micromechanical scale, which can contribute to the formation of a structural

; Goodier and Field 1963; Goodier and Kanninen 1966; Kanninen 1970] in their studies of the ductile fracture problems, where cohesive forces of nite magnitude are present at the extremities of a decohesion zone. A key featu
these models is the structural or reduced continuum representation of the decohesion zone. The lin
and nonlinear ligament models also allow for the interpretation of intermolecular and surface force
at adhesive zones [Tonck et al. 1988; Israelachvili 1992]. In this paper, we adopt the basic concepts expounded in the structural model of contact zone response and apply it to the modeling of a conta

Keywords: Winkler ligaments, contact problem, variational method, adhesive contact, compliance of indentor, rigid circular indentor, variational solution, elastic halfspace indentation.

between an isotropic elastic halfspace region and a rigid cylindrical indentor, which is achieved thro
a continuously distributed set of ligament connections. The term bonded or adhesive is avoided in
the present discussion since these speci cally refer to phenomena where complete continuity of d
placements is established at the connecting zone. In particular, we restrict attention to the modeling
the interface as a series of Winkler elements, although the approach can be extended to include m
advanced structural contacts represented by either Vlazov- and Reissner-type layers [Selvadurai 1979],
which provide shear interaction between the Winkler elements, or the constrained elastic layer, whe
certain traction boundary conditions at the edges of the ligament zones are satis ed in an integral se
A more appropriate terminology that describes this type of contact is structural bonding. An alternative
to this approach is to consider the connecting layer as an elastic continuum itself. An example of su
an application with relevance to nanorheological analysis of the contact between an elastic sphere a
a plane separated by an interfacial elastic layer is given by Trifa et al. [2002] in connection with the
compressive load transfer at a ligament zone. The Winkler ligament approach adopted here is perh
not the most all-encompassing treatment of the contact process, but it allows the incorporation of th
in uences of a material characteristic that could be attributed to the zones that generate the bondi
mechanism. In particular, the deformability characteristics of the substrate are accounted for in th
modeling.

In this paper we consider the axisymmetric problem of the contact between a rigid cylinder and a
isotropic elastic halfspace region, where the structural bonding zone corresponds to a series of clos
spaced Winkler ligaments. The conventional approach to the solution of the resulting mixed bounda
value problem is to reduce the analysis to the solution of a Fredholm integral equation of the second
which can only be solved in an approximate fashion either by reducing it to a matrix equation or thro
the introduction of a series representation of the solution or through a variational technique itself. H
we present a much simpler solution that is based on the application of a direct variational technique
variational technique has been successfully applied to the study of the mechanics of contact betwe
elastic continua and between structural elements and elastic continua [Kalker 1977; Selvadurai 1979;
1980; 1984; Karasudhi 1991]. This latter approach is a suitable approximation, in the sense that it yields
results in closed form, which can be used to establish the in uence of the idealized ligament zone in
load transfer mechanism between the rigid cylinder and the elastic halfspace as well as in the develo
of ligament adhesive stresses between the two regions.

2. The Winkler ligament contact problem

We consider the problem of a rigid circular cylinder of radius a and with a at base, which is connected
to an isotropic elastic halfspace region. The connectivity is provided by a set of Winkler elements th
establishes continuity of displacements between the rigid cylinder and the elastic halfspace (Figure 1).
The Winkler elements are characterized by a linear load-displacement relationship, although the analysis
can be easily extended to include a nonlinear Winkler model with no provision for energy dissipation
The rigid cylinder is subjected to an axisymmetric force of magnitude P which induces rigid body
displacement of the cylinder, a deformation of the set of Winkler ligaments and the displacements of the
surface of the halfspace region.

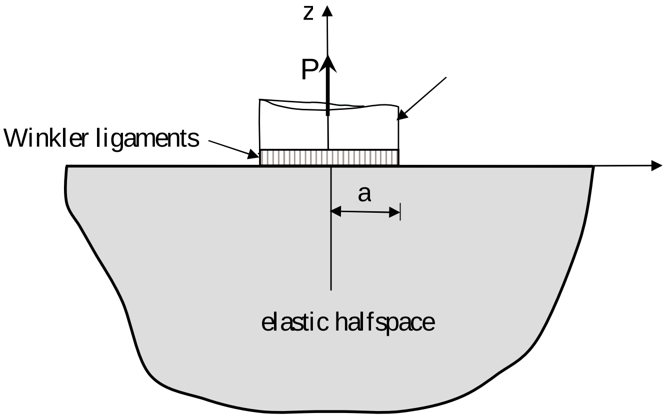

Figure 1. Contact problem for a rigid cylinder achieved through a layer of Winkler ligaments

In the variational approach adopted here, we assume that the vertical displacements of the surfa of the halfspace region, within the contact region, can be approximated by a kinematically admissib displacement of the form

$$u_z^{HS}(r,0) = a\left[C_1 - C_2\left(\frac{r}{a}\right)^2\right]; \quad r \in (0,a); \tag{1}$$

where C_1 and C_2 are arbitrary constants. Similarly, we assume that due to the loading of the rigid dis the Winkler ligaments experience a displacement

$$u_z^{W}(r,0) = a\left[C_3 + C_2\left(\frac{r}{a}\right)^2\right.$$

where u_z^{HS} is the axial displacement of the halfspace region and $_{zz}$ and $_{rz}$ are the stress components referred to the cylindrical polar coordinate system $.r; z/$. In addition, the displacements and stress elds should satisfy regularity conditions, which ensure that the displacement and stress elds decay unifo
to zero as $r; z/ ! 1$. The solution of the mix

where k is the stiffness of the Winkler ligament per unit area. The work of the applied force P is given by $W_P = -P\,[u_z^{HS}(0,0) + u_z^{W}(0,0)]$. The total potential energy function for the system can be evaluated in the form

$$U = \frac{2Ga^3}{(1-\nu)}\left[C_1^2 - \tfrac{4}{3}C_1C_2 + \tfrac{4}{5}C_2^2\right] + \frac{ka^4}{2}\left[\tfrac{1}{3}C_2^2 + C_2C_3 + C_3^2\right] - Pa\,[C_1 + C_3].$$

Considering the principle of minimum total potential energy for a conservative system, the arbitrary constants are determined from the conditions

$$\frac{\partial U}{\partial C_1} = \frac{\partial U}{\partial C_2} = \frac{\partial U}{\partial C_3} = 0$$

which gives the undetermined parameters C_1, C_2 and C_3. The constants take the forms

$$[C_1, C_2, C_3] = \frac{\tilde{P}}{(16+15\lambda)}\left[3(7+5\lambda),\; -\frac{15}{2},\; -\frac{4}{\lambda}\right]; \tag{7}$$

where $\tilde{P} = P(1-\nu)/4Ga^2$ and $\lambda = \pi ka(1-\nu)/16G$. The formal variational solution for the contact problem associated with a set of Winkler ligaments is given by (1), (2), (6) and (7). Both the state of stress within the halfspace region and within the zone of Winkler ligaments can be determined from these results in conjunction with Boussinesq's solution for the loading of a halfspace region by a concentrated normal force [Timoshenko and Goodier 1970; Davis and Selvadurai 1996; Selvadurai 2001].

3. The role of the Winkler ligament zone

An inspection of the variational solution indicates that as the relative stiffness of the Winkler ligament zone (as defined by the parameter λ) increases, the terms incorporating C_2 and C_3 will have a diminishing influence on the load transfer process. In the limit as $\lambda \to \infty$, $C_1 \to \tilde{P}$ and the displacement of the rigid cylinder is given by $w(0) = P(1-\nu)/4Ga$, and the contact stress within the circular region is

$$\sigma_{zz}(r,0) = -P/2\pi a\sqrt{a^2 - r^2};$$

which is Boussinesq's classical result for the indentation of a halfspace by a rigid circular indentor with a flat base. In terms of the contact problem, a ligament zone of high relative stiffness will invariably result in the development of a singular stress state at the boundary of the circular cylinder, which would represent a potential location for the development of delamination. For a finite value of the relative stiffness parameter λ, the displacement of the rigid cylinder as well as the stresses in the ligament zone are influenced by the Winkler ligament stiffness k. Figure 2 illustrates the variation in the normalized displacement of the rigid disc $\tilde{\Delta}$ (defined as $4G\Delta a/P(1-\nu)$, where Δ is the displacement of the rigid disc) as a function of the relative stiffness parameter λ. As can be observed, the reduction to the case of the classical Boussinesq rigid punch problem is achieved for a value $\lambda > 5$. The contact stress at the cylinder-Winkler ligament layer can similarly be evaluated in explicit form. From (6) and (7) we obtain

$$\tilde{\sigma} = \frac{\sigma_{zz}(r,0)}{\sigma_0} = \frac{1}{2(15\lambda+16)}\left[\frac{(15\lambda+21)}{\sqrt{1-\rho^2}} + 15\sqrt{1-\rho^2} - \frac{\rho^2}{\sqrt{1-\rho^2}}\right], \quad \rho \in (0,1); \tag{8}$$

Figure 2. Influence of the relative stiffness parameter on the displacement of the bonded disc

where $\sigma_0 = P/\pi a^2$ and $\rho = r/a$. Figure 3 illustrates the variation in the contact stress as a function of the relative stiffness parameter λ. As $\lambda \to 0$, the normal stresses exhibit a nonuniform distribution at the adhesive zone, but maintain the singular character, derived from the appropriate terms in (8). As $\lambda \to \infty$, the adhesive stresses reduce to the Boussinesq-type distribution, with singular behaviour as $\rho \to 1$. It is of interest to examine the influence of the relative stiffness parameter λ in moderating the stress intensity factor at the boundary of the ligament zone, which can be compared with the critical stress intensity factor necessary to initiate brittle fracture at the boundary of the adhesion zone. Considering the definition of the Mode I stress intensity factor we have

$$K_I^a = \lim_{r \to a^-} [2\pi(a - r)]^{1/2} \sigma_{zz}(r, 0). \tag{9}$$

Considering (8) and (9) we obtain

$$K_I^a = \frac{\sigma_0 \sqrt{\pi a}}{2} \frac{15\lambda + 6}{15\lambda + 16}.$$

Again as $\lambda \to \infty$, we recover from the above equation the classical result for the stress intensity factor associated with the axisymmetric problem of an elastic medium of infinite extent with an intact region of radius a and subjected to a far-field stress that is equivalent to a total force P [Kassir and Sih 1968]. Also, as $\lambda \to 0$, the stress intensity factor approaches the value $K_I^a = 3\sigma_0\sqrt{\pi a}/16$. This result is consistent with the observation made by

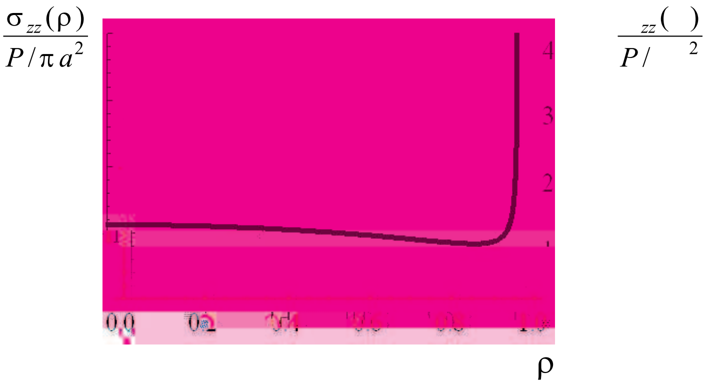

Figure 3. Adhesive stresses at the bonded zone T ! 0U (left) and T • ! 1U (right).

entail a numerical solution technique. The variational procedure provides a convenient approach f
examining the particular in uences of the Winkler ligament zone that provides the structural bonding
between the rigid cylinder and the halfspace region. The displacement functions chosen satisfy t
kinematic constraints and the range of the polynomial expressions used can be extended to include
terms. Such a treatment is perhaps unwarranted in view of the elementary nature of the modeling o
ligament zone as a continuous distribution of unconnected spring elements. The elementary analy
nonetheless illustrates trends that are important to the understanding of the mechanics of load trans
at ligament zones. The form of the displacement functions chosen for the variational treatment st
maintains the singular behaviour of the stress states in the ligament zone • 2 f0; 1 /, although such
an interpretation should be viewed with some caution, since at the outset the stiffness of the ligame
zone is assumed to be nite. In particular, it is noted that the presence of a ligament zone of low relative
stiffness has a tendency to moderate the stress intensity factor at the boundary of the ligament zone
should also be borne in mind that structural adherents with lower stiffness generally tend to posse
lower resistance to fracture, indicative of low values of the critical stress intensity factors. Finally, the
variational approach for st e

[Moreau et al. 1988] J. J. Moreau, P. D. Panagiotopoulos, and G. Strang (editors), *Topics in non-smooth mechanics*, Birkhäuser, Basel, 1988.

[Pizzi and Mittal 1994] A. Pizzi and K. L. Mittal (editors), *Handbook of adhesive technology*, Dekker, New York, 1994.

[Plueddemann 1974] E. P. Plueddemann (editor), *Interfaces in polymer matrix composites*, Composite Materials 6, Academic Press, New York, 1974.

[Selvadurai 1979] A. P. S. Selvadurai, *Elastic analysis of soil-foundation interaction*, Developments in Geotechnical Engineering 17, Elsevier, Amsterdam, 1979.

[Selvadurai 1980] A. P. S. Selvadurai, "Elastic contact between a flexible circular plate and a transversely isotropic elastic halfspace", *Int. J. Solids Struct.* 16:2 (1980), 167–176.

[Selvadurai 1984] A. P. S. Selvadurai, "A contact problem for a Reissner plate and an isotropic elastic halfspace", *J. Méc. Théor. Appl.* 3 (1984), 181–196.

[Selvadurai 2000a] A. P. S. Selvadurai, *Partial differential equations in mechanics, 2: The biharmonic equation, Poisson's equation*[Selvadurai