
Boundary element formulation of axisymmetric problems for an elastic halfspace

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elastic halfspace, given in terms of integrals of the Lipschitz...Hankel type, that satisfies in advance the boundary condition of zero traction on the free surface and the decay of displacements in the far field. Explicit equations for post-processing the results at internal points are provided, as well as adequate numerical schemes to evaluate the boundary integrals arising in the method. This formulation can be easily implemented in existing BE computational codes for axisymmetric fullspace problems, requiring only a few modifications. Numerical results are provided to validate the proposed formulation.

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1. Introduction

[19,20]

The axisymmetric formulation in classical elasticity is useful for the analysis of problems in geomechanics [1,2], as well as contact problems for cylinders, spheres and circular plates [3..8]. Other applications involve the study of fracture mechanics phenomena and inclusions [5,9..11].

In particular, the BE method is advantageous for axisymmetric problems, since it reduces the analysis of the three-dimensional domain to a one-dimensional mesh discretization requiring only the evaluation of linear integrals. However, the fundamental solutions involved are more complex, requiring special considerations on their manipulation and integration to correctly evaluate the influence coefficients arising in the boundary integral equations. Extensive surveys on the existing axisymmetric fundamental solutions are given by Wang and Liao [12,13], Wang et al. [14] and Wideberg and Benitez [15].

The BE method for axisymmetric elasticity was first formulated by Cruse et al. [16], using the fullspace fundamental solution derived by Kermanidis [17]. Several contributions to the formulation of the axisymmetric problem may be cited, such as the expansion of non-symmetric boundary conditions by Fourier series suggested by Mayr [18] and Rizzo and Shippy

the number of unknown functions can be reduced from 12 to 6. These functions can be determined using (i) the compatibility conditions of displacements at the interface

u_i^+

$$u_{zr}^{nd} \frac{1}{4} \frac{1}{\text{---}}$$

where G_E and \bar{G}_E are portions of the circumference of radius E as depicted in Fig. 4.

za 0. On the other hand, for z¼0 the traction term t_{ij}nd vanishes and only u_{ij}nd presents singularity.

In this section, the integration cases identi“ed for G_{ia}^t and H_{ia}^t are grouped according to the position of P (x,z) in relation to the part of the boundary along which the integration is carried out as well as to the axis of symmetry. The numerical schemes employed in this work to evaluate regular integrals, weakly singular integrals of logarithmic terms and the finite part of singular integrals of order 1=r are briefly presented in [Appendix C](#).

4.1. Case 1: P(x,z) 2 G

If the point P (x,z) does not belong to the portion of the boundary being integrated, as illustrated in [Fig. 5](#), then r > 0 and accordingly both G_{ia}^t and H_{ia}^t expressed in Eqs. (56) and (57) are regular and can be evaluated by the Gauss-Legendre quadrature rule [65]. For each portion of the boundary, these integrals can be rewritten in terms of the natural coordinate Z ½ 1; 1 as

$$G_{ia}^t \frac{1}{4\pi} \int_{-1}^1 u_{ij}^{nh} N_a \delta_{ij} \frac{dZ}{\sqrt{1-Z^2}} \quad (58)$$

$$H_{ia}^t \frac{1}{4\pi} \int_{-1}^1 t_{ij}^{nh} N_a \delta_{ij} \frac{dZ}{\sqrt{1-Z^2}} \quad (59)$$

where δ_{ij}

The complete elliptic integrals can be approximated as in Eqs. (62) and (63). Similar to the procedure presented for Case 2.1, one arrives at

$$G_a^{\pm} \approx \frac{Z_1}{2p_1} u$$

$$\frac{2ck^5}{\rho k^4 A_1^5} \cdot 6\delta^2 \cdot 2\rho \cdot \frac{c^2 k^2}{A_1^2} \cdot \frac{8k^4}{k^2} \cdot 23 \cdot \frac{!}{\#} \text{ E\ddot{a}m\ddot{a} } \quad \text{\AA:11\ddot{a}}$$

$$I_{113} \cdot \frac{1}{4} \cdot \frac{2ck^3}{\rho k^2 A_1^5} \cdot 3\delta^2 \cdot 2\rho \cdot \frac{2c^2 k^2}{A_1^2} \cdot \frac{2k^4}{k^2} \cdot 3 \cdot \frac{!}{\#} \text{ K\ddot{a}m\ddot{a} } \\ \frac{2ck^3}{\rho k^4 A_1^5} \cdot 46\delta^4 \cdot \rho \cdot k^2 \cdot \rho \cdot \frac{c^2 k^2 \delta^2}{k^2 A_1^2} \cdot 2\rho \delta k^2 \cdot \rho \cdot 8k^4 \cdot \frac{3}{5} \text{ E\ddot{a}m\ddot{a} } \quad \text{\AA:12\ddot{a}}$$

$\bar{u}_{zzr}^{nf} \frac{1}{8p\delta l} \frac{1}{nb} P_2 \bar{I}_{011} p \bar{z} \bar{g}_{012} g$	8:6P	$S_{rrz}^{nd} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{x} \frac{1}{2} \text{sign} \bar{z} P_1 \bar{I}_{111} p \delta P_5 z p z \bar{H}_{112}$	
$\bar{u}_{zzz}^{nf} \frac{1}{8p\delta l} \frac{1}{nb} \text{sign} \bar{z} P_1 \bar{I}_{001} \bar{z} \bar{I}_{002} g$	8:7P	$\text{sign} \bar{z} P_1 \bar{I}_{113} p \text{sign} \bar{z} P_1 \bar{I}_{012} \delta 3z p z \bar{H}_{013}$	8:24P
$\bar{u}_{rrr}^{nd} \frac{1}{8p\delta l} \frac{1}{nb} \frac{1}{x} \frac{1}{2} P_7 \bar{I}_{110} p P_5 \bar{z} \bar{g}_{111} 2zz \bar{I}_{112} p P_6 \bar{I}_{011}$ $\text{sign} \bar{z} P_5 z^0 p 3z \bar{H}_{012} p 2zz \bar{I}_{013}$	8:8P	—	
$\bar{u}_{rrz}^{nd} \frac{1}{8p\delta l} \frac{1}{nb} \frac{1}{x} \text{sign} \bar{z} P_1 P_2 \bar{I}_{100} P_5 \bar{z} \bar{I}_{101}$ $\text{sign} \bar{z} P_2 \bar{I}_{102} \text{sign} \bar{z} P_3 \bar{I}_{001} p \delta P_5 z^0 3z \bar{H}_{002}$ $p \text{sign} \bar{z} P_2 \bar{I}_{003}$	8:9P		
$\bar{u}_{rrz}^{nd} \frac{1}{8p\delta l} \frac{1}{nb} \text{sign} \bar{z} P_1 \bar{I}_{111} p \delta P_5 z^0 p z \bar{H}_{112} \text{sign} \bar{z} P_2 \bar{I}_{113} g$	8:10P		
$\bar{u}_{rzz}^{nd} \frac{1}{8p\delta l} \frac{1}{nb} P_2 \bar{I}_{101} \text{sign} \bar{z} P_5 z^0 z \bar{H}_{102} 2zz \bar{I}_{103} g$	8:11P		
$\bar{u}_{zzr}^{nd} \frac{1}{8p\delta l} \frac{1}{nb} P_2 \bar{I}_{011} p \text{sign} \bar{z} P_5 z^0 z \bar{H}_{012} 2zz \bar{I}_{013} g$	8:12P		
$\bar{u}_{zzz}^{nd} \frac{1}{8p\delta l} \frac{1}{nb} \text{sign} \bar{z} P_1 \bar{I}_{001} \delta P_5 z^0 p z \bar{H}_{002}$ $2 \text{sign} \bar{z} P_1 \bar{I}_{003} g$	8:13P		
$S_{rrr}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{x} \frac{1}{2} P_4 \bar{I}_{101} \bar{z} \bar{g}_{102} p \frac{1}{xr} \frac{1}{2} P_5 \bar{I}_{110} p \bar{z} \bar{g}_{111}$ $p \frac{1}{r} P_4 \bar{I}_{011} \bar{z} \bar{g}_{012} 3 \bar{I}_{002} \bar{z} \bar{g}_{003}$	8:14P		
$S_{rrz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{x} \text{sign} \bar{z} P_1 \bar{I}_{111} \bar{z} \bar{I}_{112} \text{sign} \bar{z} P_1 \bar{I}_{012} p \bar{z} \bar{I}_{013}$	8:15P		
$S_{rrzz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{x} \frac{1}{2} P_2 \bar{I}_{101} p \bar{z} \bar{g}_{102} p \bar{I}_{002} \bar{z} \bar{g}_{003}$	8:16P		
$S_{rzz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{r} \frac{1}{2} \text{sign} \bar{z} P_1 \bar{I}_{111} p \bar{z} \bar{I}_{112} p \text{sign} \bar{z} P_1 \bar{I}_{012} \bar{z} \bar{I}_{103}$	8:17P		
$S_{rzz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \bar{I}_{112} \bar{z} \bar{g}_{113} g$	8:18P		
$S_{rzzz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \bar{z} \bar{I}_{103}$	8:19P		
$S_{zzrr}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{r} \frac{1}{2} P_2 \bar{I}_{011} p \bar{z} \bar{g}_{012} p \bar{I}_{002} \bar{z} \bar{g}_{003}$	8:20P		
$S_{zzrz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \bar{z} \bar{I}_{013}$	8:21P		
$S_{zzzz}^{nf} \frac{1}{4p\delta l} \frac{m}{nb} \bar{I}_{002} p \bar{z} \bar{g}_{003} g$	8:22P		
$S_{rrr}^{nd} \frac{1}{4p\delta l} \frac{m}{nb} \frac{1}{x} P_6 \bar{I}_{101} \text{sign} \bar{z} P_5 z p 3z \bar{H}_{102}$ $p 2zz \bar{I}_{103} p \frac{1}{xr} \frac{1}{2} P_7 \bar{I}_{110} 2zz \bar{I}_{112} p P_5 \bar{z} \bar{g}_{111}$ $p \frac{1}{r} P_6 \bar{I}_{011} \text{sign} \bar{z} P_5 z^0 p 3z \bar{H}_{012} p 2zz \bar{I}_{013}$ $5 \bar{I}_{002} p 3z \bar{g}_{003} 2zz \bar{I}_{004}$	8:23P		

n

where

$$J = \frac{dr}{dz} = \frac{dz}{dr} \quad (2)$$

is the Jacobian transformation between the global and natural coordinate systems. The coefficients Z_m^g and w_m^g are the abscissas and weights of the Gauss-Legendre quadrature rule for n_g points within the interval $[-1; 1]$ which suffice to exactly evaluate the integral of a polynomial of order $2n_g - 1$.

C.2. Weakly singular integral of logarithmic terms

Let $f(z)$ be a regular function and

where \bar{m} is given by Eq. (C.8) and

$$E_1 \bar{m}^4 = a_1 \bar{m}^3 + b_1 a_4 \bar{m}^4 \quad (C.19)$$

$$E_2 \bar{m}^4 = a_2 \bar{m}^3 + b_2 a_4 \bar{m}^4$$

are the polynomials whose coefficients are given by

$$a_1 = 0.44325141463, \quad b_1 = 0.24998368310$$

$$a_2 = 0.06260601220, \quad b_2 = 0.09200180037$$

$$a_3 = 0.04757383546, \quad b_3 = 0.04069697526$$

$$a_4 = 0.01736506451, \quad b_4 = 0.00526449639 \quad (C.20)$$

The polynomial approximation of $E(m)$ presents no singularity, since $E_2 \bar{m}^4$ has no free coefficients, according to Eq. (C.18). However, the presence of $\ln \bar{m}$ causes the integrand of Eq. (C.17) to be non-analytical, which requires a special numerical treatment.

In a manner similar to that used in the previous section, the following expression can be obtained for the numerical evaluation of the weakly singular integral given by Eq. (C.17)

$$\int_G f(\bar{\alpha}, z) E_1 \bar{m}^4 dG = \sum_{m=1}^{X_g} \int_G f(\bar{\alpha}, z) E_1 \bar{m}^3 + b_1 E_2 \bar{m}^4 \ln \frac{1}{2} \frac{Z}{\bar{m}} dG + \sum_{m=1}^{X_g} w_m^g \quad (C.21)$$

$$4 \sum_{m=1}^{X_g} \int_G f(\bar{\alpha}, z) E_2 \bar{m}^4 \ln \frac{Z}{2\bar{m}} dG + \sum_{m=1}^{X_g} w_m^l \quad (C.22)$$

for Z given by Eq. (C.6).

C.3. Cauchy principal value of the singular integral of order 1 = r

Let $f(\bar{\alpha}, z)$ be a regular function and $r(\bar{\alpha}, z)$ the distance between the points $P(\bar{\alpha}, z)$ and $Q(\bar{\alpha}, z)$ on the boundary $G(\bar{\alpha}, z)$. The strongly singular integral

$$\int_G \frac{f(\bar{\alpha}, z) dz}{r(\bar{\alpha}, z)} \quad (C.23)$$

has to be evaluated for the case $r \rightarrow 0$ or $r \rightarrow \infty$. This integral may be obtained as a sum of a Cauchy principal value and a discontinuous term as

$$\int_G \frac{f(\bar{\alpha}, z) dz}{r(\bar{\alpha}, z)} = PV \int_G \frac{f(\bar{\alpha}, z) dz}{r(\bar{\alpha}, z)} + c \quad (C.24)$$

The evaluation of the discontinuous term c of the strongly singular integrals appearing in the boundary element formulations is addressed in Section 3.2.

The Cauchy principal value is best evaluated in terms of two "finite-part integrals, denoted by \mathcal{P} , for the boundary segments adjacent to the singularity point $r(\bar{\alpha}, z) \rightarrow 0$.

In what follows, the integration scheme proposed by Dumont and Souza [67] is used. Using the notation of Eq. (C.6), the regular function can be expanded as a Taylor series to obtain the following normalized integral of Eq. (C.23) over the curved boundary G

$$\int_G \frac{f(\bar{\alpha}, z) dz}{r(\bar{\alpha}, z)} = \sum_{m=0}^{X_g} \frac{f_m(\bar{\alpha}, z)}{r(\bar{\alpha}, z)} dZ + \sum_{m=1}^{X_g} w_m^g \quad (C.25)$$

The resulting quadrature rule for evaluating Cauchy's principal value of the strongly singular integral of (C.23) is given by

$$\int_G \frac{f(\bar{\alpha}, z) dz}{r(\bar{\alpha}, z)} = \sum_{m=1}^{X_g} \int_G \frac{f_m(\bar{\alpha}, z) dZ}{r(\bar{\alpha}, z)} + \sum_{m=1}^{X_g} w_m^g$$

$$\sum_{m=1}^{X_g} \int_G \frac{f_m(\bar{\alpha}, z) dZ}{r(\bar{\alpha}, z)} + \sum_{m=1}^{X_g} w_m^g \quad (C.26)$$

where

$$\int_G \frac{f_m(\bar{\alpha}, z) dZ}{r(\bar{\alpha}, z)} = \int_G \frac{f_m(\bar{\alpha}, z) dZ}{r(\bar{\alpha}, z)} \quad (C.27)$$

The above scheme, that employs the Gauss-Legendre quadrature rule and an additional correction term, evaluates exactly this integral for a polynomial function of order $2 - n_g$. Other numerical integration schemes for the strongly singular integral can be used [72,73].

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