McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE MATHEMATICS
Paper BETA

19 August 2016
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an e ort to ensure that your arguments are

Algebra Module

[ALG. 1]

Analysis Module

[AN. 1]

Let $\varepsilon > 0$ be fixed. Show that the set of all real numbers $x \in [0,1]$ such that there exist infinitely many pairs $p, q \in \mathbb{N}$ such that $|x - p/q| < 1/q^{2+\varepsilon}$ has Lebesgue measure 0.

[AN. 2]

Let f be a uniformly continuous function on $\mathbb{R}$. Suppose that $f \in L^p$ for some p, $1 \le p < \infty$. Prove that $f(x) \to 0$ as $|x| \to \infty$.

[AN. 3]

(a) Give a definition of $||f||_{\infty}$ of a measurable complex function f.

(b) Recall that the essential range of a function $f \in L^{\infty}(\Omega;\mathbb{C})$ is the set consisting of complex numbers w such that

$$\mu(\{x : |f(x) - w| < \varepsilon\}) > 0$$

for every $\varepsilon > 0$. Prove that R_f is compact.

(c) Show that $||f||_{\infty} = \sup_{w \in R_f} |w|$.

[AN. 4]

(a) Give a definition of a locally compact topological space.

(b) Give an example of a Borel measure μ on $\mathbb{R}$ such that $X = L^2(\mathbb{R};\mu)$ is locally compact and explain why it is so.

(c) Give an example of a Borel measure μ on $\mathbb{R}$ such that $X = L^2(\mathbb{R};\mu)$ is not locally compact and explain why it is so.

Geometry and Topology Module

[GT. 1]

(a) Suppose that X is a separable metric space. Show that any subspace of X is separable.

(b) Suppose that X is a compact metric space. Show that X is separable and that any compatible metric on X is complete.

[GT. 2]

(a) Show that the connected sum $T\#P$ of the torus T and the projective plane P is homeomorphic to the connected sum of three copies of the projective plane $P\#P\#P$.

(b) The boundary of the Möbius band is a circle. Which surface do we obtain if we identify antipodal points of that circle? Justify your answer.

[GT. 3]

Let G be a Lie group acting on a manifold M transitively, let H be a connected compact Lie subgroup of G which is an isotropic group of a point $p \in M$. Show that M has a Riemannian metric such that the transformation determined by each element of G is an isometry.

[GT. 4]

Let M be a Riemannian manifold of dimension n and let $p \in M$. Prove that there is a neighborhood U of p and n vector fields $e_1, \dots, e_n$ in U, such that

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad \nabla_{e_i} e_j(p) = 0, \quad \forall i, j = 1, \dots, n.$$